Pre-class Warm-up!!! ☃️

One of the following functions is a solution to the differential equation

$$y'' - 3y' + 2y = e^{3x}$$

Just by guessing what seems reasonable, which function do you think it probably is?

a. $y = \sin 3x$

b. $y = 1 - 3x + 2x^2$

c. $y = \sin x \, e^{3x}$

d. $y = \frac{1}{2} e^{3x}$ ✓

Section 5.5: Non-homogeneous equations

We learn:

- how to find a particular solution to an equation

$a_n y^{(n)} + \ldots + a_0 y = f(x)$

where $f(x)$ is a function made up of polynomials, exponentials and sin, cos

- the method of undetermined coefficients
- The method of variation of parameters

We know already:

- the general solution is then $y_c + y_p$ where
- y_p is a particular solution, and
- y_c is a solution to the corresponding homogeneous equation

Question like 5.5: 1-20

Find a particular solution to the equation

$$y'' - 3y' + 2y = e^{3x}$$

Solution: We guess a solution $y = Ae^{3x}$

$y' = 3Ae^{3x}$ $y'' = 9Ae^{3x}$. Substitute

$$(9A - 9A + 2A)e^{3x} = e^{3x}$$

$$2A = 1$$

$$A = \frac{1}{2}.$$

Like questions 31-40

Solve the initial value problem

$$y'' - 3y' + 2y = e^{3x}, \quad y(0) = 2\tfrac{1}{2}, \quad y'(0) = 4\tfrac{1}{2}$$

Solution. Step 1: find a particular solution

y_p = (1/2) e^3x Done!

Step 2: find a general complementary function

y_c = A y_1 + B y_2

Step 3: Apply the initial conditions to

y = y_p + y_c to find A and B.

Step 2: solve the homogeneous equation

y'' - 3y' + 2y = 0

Characteristic equation r^2 - 3r + 2 = 0,

(r-1)(r-2) = 0, r = 1 or 2 $y_c = Ae^x + Be^{2x}$

Step 3: Apply initial conditions to

$$y = y_p + y_c = \frac{1}{2}e^{3x} + Ae^x + Be^{2x}$$

$$y(0) = \frac{1}{2} + A + B = 2\tfrac{1}{2}$$

$$y' = \frac{3}{2}e^{3x} + Ae^x + 2Be^{2x}, \quad y'(0) = \frac{3}{2} + A + 2B = 4\tfrac{1}{2}$$

$$A + B = 2$$
$$A + 2B = 3$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y = \frac{1}{2}e^{3x} + e^x + e^{2x}$$

Question like 5.5: 1-20

How would you do: ?

Find a particular solution to the equation

$$y'' - 3y' + 2y = e^{3x} \qquad \text{Try } y = Ae^{3x}$$

$$2x + 3x^2 \qquad y = A + Bx + Cx^2$$

$$\sin x \qquad y = A\sin x + B\cos x$$

$$x^2 e^{3x} \qquad y = Ax^2e^{3x} + Bxe^{3x} + Ce^{3x}$$

$$\sin x \, e^{3x} \qquad y = A\sin x\, e^{3x} + B\cos x\, e^{3x}$$

$$x\sin x \qquad y = Ax\sin x + Bx\cos x + C\sin x + D\cos x$$

$$e^{x} \qquad y = Ae^{x}$$

What functions should we try to get a solution to the non-homogeneous equation?

Question like 5.5: 1-20

Find a particular solution to the equation

$y'' - 3y' + 2y = e^x$

Attempt: Try $y = Ae^x$, $y' = Ae^x = y''$.

Substitute: $Ae^x - 3Ae^x + 2Ae^x = e^x$

$0 = e^x$

Solution: Try $y = Axe^x \underbrace{+ Be^x}_{\text{not needed}}$

$y' = A(e^x + xe^x)$ $y'' = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x$

Substitute:

$2Ae^x + Axe^x - 3Ae^x - 3Axe^x + 2Axe^x = -Ae^x = e^x$

Thus $A = -1$

$y = -xe^x$ is a particular solution.

The problem with trying $y = Ae^x$ is that e^x is a solution to the corresponding homogeneous equation, so we get 0 when we substitute it into the left side.

The answer is to try $y = Axe^x$ instead.

But what if both e^x and xe^x are solutions of the homogeneous equation?

Pre-class Warm-up!!!

If we write $\cos 3t - 2\sin 3t = C\cos(\omega t - \alpha)$

what is $\tan\alpha$?

a. -1/2
b. 4
c. 2
d. -2 ✓
e None of the above.

What is C ?

a. 5
b. √5 ✓
c. √3
d. -1
e. None of the above

On Friday March 29 this class will be held in Murphy Hall 130

On Monday April 1, I will teach this class remotely, on Zoom, with a link I will send you.

The quiz tomorrow is on sections 4.7-5.4
4.7 is about abstract vector spaces.

Question like 5.5: 21-30

What is the appropriate form of a particular solution to the following equation?

$$y^{(5)} + y^{(3)} = 1+2x+3x^2$$

Solution: Char polynomial $r^5+r^3 = r^3(r^2+1)$ has roots 0 (3 times), $\pm i$.

General solution: $A+Bx+Cx^2+D\cos t+E\sin t$ to the homogeneous equation.

For a solution to $\cdots = 1+2x+3x^2$ we try

$y = Ax^3+Bx^4+Cx^5$. Here $5 = 2+3$ (exp of x^2, times repeated)

Substitute: $C\cdot 120 + 6A + 24Bx + 60Cx^2 = 1+2x+3x^2$

Coeff of x^2: $60C = 3$, $C = \frac{1}{20}$

x: $24B = 2$, $B = \frac{1}{12}$

1: $6A = -5$, $A = -5/6$

$y = -\frac{5}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{20}x^5$ is a particular solution.

What about

$y^{(5)} + y^{(3)} = \sin x$ — b. works — a solution of homog. eqn.

$x\sin x$ — d. works, b. doesn't work.

$e^x \sin x$ — $Ae^x\sin x + Be^x\cos x$?

To get a particular solution, we should try

a. $A\sin x + B\cos x$

b. $Ax\sin x + Bx\cos x$

c. $Ax^2\sin x + Bx^2\cos x$

d. $Ax\sin x + Bx\cos x + Cx^2\sin x + Dx^2\cos x$

e. None of the above.

Question like 5.5, 47-56.
Use the method of variation of parameters to find a particular solution to the equation.

51. $y'' + 4y = \cos 3x$

Idea. Char poly is r^2+4, roots $\pm 2i$

General soln for homog. eqn: $A\cos 2x + B\sin 2x$

We try a particular solution of form

$y = u\cos 2x + v\sin 2x$ where u, v are functions of x.

Calculate $y' = u(-2\sin 2x) + v(2\cos 2x)$
$+ u'\cos 2x + v'\sin 2x$.

Idea: put $u'\cos 2x + v'\sin 2x = 0$

Next $y'' = u(\quad) + v(\quad)$
$+ u'(-2\sin 2x) + v'(2\cos 2x)$

Substitute in $y'' + 4y = \cos 3x$.

All terms disappear except for the terms in u', v'.

Get two equations for u', v'.

Solve. Do $u = \int u'$ $v = \int v'$.

Question like 5.5, 47-56.
Use the method of variation of parameters to find a particular solution to the equation.

51. $y'' + 4y = \cos 3x$

Solution: We have solutions $\cos 2x$, $\sin 2x$ to $y'' + 4y = 0$.
Try for a solution $y = u \cos 2x + v \sin 2x$ where u, v are functions of x. Then
$y' = -2u \sin 2x + 2v \cos 2x + u' \cos 2x + v' \sin 2x$
It works if we put the pink terms
$u' \cos 2x + v' \sin 2x = 0$
Next
$y'' = -4u\cos 2x - 4v\sin 2x - 2u'\sin 2x + 2v'\cos 2x$
Substitute in the original equation. Some cancellation takes place (no accident):
$y'' + 4y = - 2u'\sin 2x + 2v'\cos 2x = \cos 3x$

We get a system of linear equations for u', v'

$$\begin{bmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ \cos 3x \end{bmatrix}$$

$$\overset{\text{"}}{W}(\cos 2x, \sin 2x)$$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2\cos 2x & -\sin 2x \\ 2\sin 2x & \cos 2x \end{bmatrix} \begin{bmatrix} 0 \\ \cos 3x \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -\sin 2x \cos 3x \\ \cos 2x \cos 3x \end{bmatrix}$$

$$u = -\frac{1}{2} \int \sin 2x \cos 3x \, dx = -\frac{1}{2} \int (\sin 5x - \sin x) dx$$

$$= \frac{1}{10} \cos 5x - \frac{1}{2} \cos x$$

$$v = \frac{1}{2} \int \cos 2x \cos 3x \, dx = \frac{1}{2} \int (\cos 5x + \cos x) dx$$

$$= \frac{1}{10} \sin 5x + \frac{1}{2} \sin x$$

$$y = \left(\frac{1}{10} \cos 5x - \frac{1}{2} \cos x\right) \cos 2x + \left(\frac{1}{10} \sin 5x + \frac{1}{2} \sin x\right) \sin 2x$$

Question:
What is the best form of function to try in solving
$y'' + y = x \sin x$?

a. $ax \sin x + bx \cos x$

b. $ax \sin x + bx \cos x + c \sin x + d \cos x$

c. $ax \sin x + bx \cos x + cx^2 \sin x + dx^2 \cos x$ ✓

d. $a \sin x + b \cos x + cx^2 \sin x + dx^2 \cos x$

e. None of the above.

Note that $\sin x$ and $\cos x$ are solutions to
$y'' + y = 0$.

Worked example:

Find a particular solution for

$y'' + y = x \sin x$

Solution: observe $y_c = \sin x$ and $y_c = \cos x$ solve $y'' + y = 0$.

Try these cancel

$$y = x^2(a\sin x + b\cos x) + x(c\sin x + d\cos x)$$

$$y' = x^2(a\cos x - b\sin x) + 2x(a\sin x + b\cos x) + x(c\cos x - d\sin x) + c\sin x + d\cos x$$

$$y'' = x^2(-a\sin x - b\cos x) + 4x(a\cos x - b\sin x) + 2(a\sin x + b\cos x) + x(-c\sin x - d\cos x) + 2(c\cos x - d\sin x)$$

$$y'' + y = 4x(a\cos x - b\sin x) + (2c+2b)\cos x + (2a-2d)\sin x$$

$$= x\sin x$$

Thus $-4b = 1$ $\quad a = 0 = d = 2c+2b$

$b = -\frac{1}{4}$ $\quad c = +\frac{1}{4}$

$$y_p = -\frac{1}{4}x^2\cos x + \frac{1}{4}x\sin x$$